Separation and isoperimetric profiles

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Introduction

Let X and Y be two metric spaces.

How can you tell if X can embed in Y?

YES : find such an embedding

NO : find an obstruction, *e.g.* computing a **monotone invariant**.

Framework

Metric spaces X and Y: **bounded degree graphs**.

Embeddings: **regular maps**.

Definition

A map $f: X \to Y$ is called **regular** if there exists $\kappa > 0$ such that

(i) $d(f(x), f(x')) \leq \kappa d(x, x')$, for every $x, x' \in X$,

(ii) and $|f^{-1}(\{y\})| \leq \kappa$, for every $y \in Y$.

Examples of regular maps: quasi-isometric embeddings, coarse embeddings.

Separation profile

The monotone invariant that we consider here is the **separation profile**.

Definition (Benjamini, Schramm & Timár 2011, Hume 2017)

Let X be a graph. We define the **separation profile** of X as

$$\mathrm{sep}_X(n) = \sup_{\Gamma \subset X, |V\Gamma| \leq n} |V\Gamma| \, h(\Gamma),$$

where $h(\Gamma) = \inf_{A \sqcup B = V\Gamma} \frac{|E(A,B)|}{\min(|A|,|B|)}$ is the Cheeger constant of the graph Γ.

Separation profile

The separation profile is monotone under regular maps:

Proposition (Benjamini, Schramm & Timár 2011)

Let $f : X \to Y$ be a κ-regular map. Then, there exists $C = C(d_X, d_Y, \kappa)$ such that we have

$$\mathrm{sep}_X(n) \leq C\,\mathrm{sep}_Y(n), \quad \text{for any large enough } n.$$

Other known monotone invariants: asymptotic dimension, growth.

Separation profile: examples

We give few examples:

- A tree T of degree bounded by D: $\text{sep}_T(n) \leq 4D$ (Jordan)
- A d-dimensional grid $\mathbb{Z}^d$: $\text{sep}_{\mathbb{Z}^d}(n) \simeq n^{\frac{d-1}{d}}$ (BST) , also true for nilpotent groups of growth rate d (Hume, Mackay & Tessera).
- The hyperbolic space $\mathbb{H}^d$: $\text{sep}_{\mathbb{H}^2}(n) \simeq \log n$, and $\text{sep}_{\mathbb{H}^{d+1}}(n) \simeq n^{\frac{d-1}{d}}$ if $d \geq 2$ (BST).
- A graph Γ containing an expander (*i.e.* $\exists \Gamma_m \subset \Gamma$ such that $\inf h(\Gamma_m) > 0$ and $\sup |\Gamma_m| = \infty$): $\frac{\text{sep}_\Gamma(n)}{n} \not\to 0$ (Hume).

Conclusion:
The separation profile disrupts the usual hierarchy on groups.

Isoperimetric profile

Theorem (Følner)

A finitely generated group G is amenable if and only if there exists a sequence $F_n \subset G$ such that $\lim_{n\to\infty} \frac{|\partial F_n|}{|F_n|} = 0$.

Definition

For any graph G, we define the isoperimetric profile of G as:

$$\Lambda_G(n) = \inf \left\{ \frac{|\partial F|}{|F|} : F \subset VG,\ |F| \leq n \right\}$$

Decay of Λ_G $\Leftrightarrow$ Quality of the amenability of G

Isoperimetric profile: examples

Isoperimetric profiles have been well studied. Few examples:

- If F is a free group, $\Lambda_F(n) \simeq 1$.
- For the d-dimensional grid $\mathbb{Z}^d$, $\Lambda_{\mathbb{Z}^d}(n) \simeq n^{-1/d}$, as for any nilpotent group of growth rate d.

If L and S are two groups, we define their **wreath product** as the group $L \wr S = \oplus_S L \rtimes S$.

- $\Lambda_{\mathbb{Z}_2 \wr \mathbb{Z}} \simeq (\log n)^{-1}$,
- $\Lambda_{\mathbb{Z}_2 \wr \mathbb{Z}^d} \simeq (\log n)^{-1/d}$,
- $\Lambda_{\mathbb{Z}_2(\ldots(\mathbb{Z}_2(\wr\mathbb{Z}^d)\ldots)} \simeq (\log \ldots \log n)^{-1/d}$. (Erschler)

Comparing those profiles ?

Recall

$$\mathrm{sep}_G(n) = \sup_{\Gamma \subset G,\, |V\Gamma| \leq n} \left(|V\Gamma| \times \inf_{A \subset V\Gamma,\, |A| \leq \frac{|V\Gamma|}{2}} \frac{|\partial_\Gamma A|}{|A|} \right),$$

and

$$\Lambda_G(n) = \inf_{A \subset VG,\, |A| \leq \frac{|VG|}{2}} \frac{|\partial_G A|}{|A|}.$$

It is reasonable to compare $\mathrm{sep}_G(n)/n$ with $\Lambda_G(n)$.

Main theorem

Definition

Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ a non-decreasing function, satisfying $\lim_{x\to\infty} f(x) = 0$. For every $\delta \in (0,1)$, we define the δ-**geometric decay function** of f as: $p_f^\delta(t) = \min\{t' \mid f(t') \leq \delta f(t)\}$.

Theorem (Gournay, L.)

Let G be a bounded degree, amenable, connected, infinite graph. Then, for every $n \geq 1$, there exists $N \in \left[n, p_\Lambda^{1/4}(n)\right]$ such that

$$\frac{\operatorname{sep}(N)}{N} \geq \frac{1}{8} \frac{\Lambda(n)}{\log\left(\frac{p_\Lambda^{1/4}(n)}{n}\right) + 1}$$

Applications

- If G is nilpotent of growth rate d, then we recall the we have $\Lambda_G(n) \simeq n^{-1/d}$. Then, we recover from the theorem:

$$\mathrm{sep}_G(n) \succeq n^{\frac{d-1}{d}}.$$

- If G satisfies $\Lambda(n) \preceq (\log n)^{-\alpha}$ (for example $\mathbb{Z}_2 \wr \mathbb{Z}$), then we deduce:

$$\frac{\mathrm{sep}_G(N)}{N} \succeq \frac{\Lambda(N)}{\log N}, \quad \text{for infinitely many } N\text{'s.}$$

- If G is solvable with exponential growth, then there exists $\alpha \geq 1$ such that $(\log n)^{-1} \preceq \Lambda_G(n) \preceq (\log \dots \log n)^{-1/\alpha}$ (Coulhon Saloff-Coste, Saloff-Coste Zheng).
  We deduce that for every $\epsilon > 0$, we have $\text{sep}_G(N) \succeq N^{1-\epsilon}$, for infinitely many N's.

Theorem

Let G be a finitely generated solvable group. If there exist $c, \epsilon > 0$ such that $\text{sep}_G(n) \leq cn^{1-\epsilon}$ for any large enough n, then G is virtually nilpotent.

Corollary (Hume & Sisto 2017)

Let G be a solvable group such that there exists a regular map from G to an hyperbolic space $\mathbb{H}^d$. Then, G is virtually nilpotent.

Sketch of proof

Let G be a bounded degree, amenable, connected, infinite graph.

Lemma

Let F be a isoperimetric optimal subset of VG (i.e.: for any $F' \subset VG$ satisfying $|F'| \leq |F|$, we have $\frac{|\partial F'|}{|F'|} \geq \frac{|\partial F|}{|F|}$). Then,

$$2h(F) \geq \Lambda_G\left(\frac{|F|}{2}\right) - \Lambda_G(|F|)$$

Corollary

Let n be such that there exists an isoperimetric optimal subset F of cardinality n. Then,

$$2\frac{\text{sep}_G(n)}{n} \geq \Lambda_G(n/2) - \Lambda_G(n)$$

(we call such an integer "optimal")

Proof.

Let $A \sqcup B = F$ be a partition of F, with $|A| \leq \frac{|F|}{2}$. We have $\partial_F A = E(A,B) = \partial_G A \cap \partial_G B$, and then

$$|\partial_G F| = |\partial_G A| + |\partial_G B| - 2\,|\partial_F A|\,.$$

From this, we deduce:

$$\begin{aligned} 2\,|\partial_F A| &= |\partial_G A| + |\partial_G B| - |\partial_G F| \\ &\geq \Lambda\left(\frac{|F|}{2}\right)|A| + |B|\,\frac{|\partial_G F|}{|F|} - |F|\,\frac{|\partial_G F|}{|F|} \\ &= \Lambda\left(\frac{|F|}{2}\right)|A| - \Lambda(|F|)\,|A| \end{aligned}$$

□

Sketch of proof of the theorem

To simplify, we imagine that every integer is isoperimetric optimal. Consider a positive integer n, and a sequence of integers $n_0 = n$, n_1, ... ,$n_{i_{max}}$ like this:

$$n_0 = n$$
$$n_1 = 2n_0$$
$$n_2 = 2n_1$$
$$\dots$$

$n_{i_{max}}$ satisfying $\Lambda(n_{i_{max}}) = \Lambda(n)/2$. $(\Leftrightarrow n_{i_{max}} = p^{1/2}_{\Lambda_G}(n))$

Then,

$$\sum_{i=1}^{i_{max}} \Lambda(n_i/2) - \Lambda(n_i) = \Lambda(n) - \Lambda(p^{1/2}_{\Lambda_G}) = \frac{1}{2}\Lambda(n).$$

Sketch of proof of the theorem

Then we have, from the lemma,

$$\sum_{i=1}^{i_{max}} \frac{\text{sep}_G(n_i)}{n_i} \geq \frac{1}{2}\Lambda(n).$$

Hence, there exists i such that

$$\frac{\text{sep}_G(n_i)}{n_i} \geq \frac{1}{2}\frac{\Lambda(n)}{i_{max}} \geq \frac{1}{2}\frac{\Lambda(n)}{\log\left(p_{\Lambda_G}^{1/2}(n)/n\right)}.$$

A local approach: definition

The proof is local. It motivated us to give the following definition.

Definition

Let G be a graph. We define the local separation profile at a vertex v of G as

$$\mathrm{sep}_G^v(n) = \sup\left\{|F|\, h(F) \colon F \subset B_G(v,r),\ |B_G(v,r)| \leq n\right\}$$

Application to percolation clusters

Using a result of Pete concerning local isoperimetry of percolation clusters in $\mathbb{Z}^d$, we obtained the following theorem:

Theorem

Let $p > p_c(\mathbb{Z}^d)$. Let ω be percolation configuration of $\mathbb{Z}^d$ of parameter p. Let $\mathcal{C}_\infty$ be "the" infinite connected component of ω. Let $\epsilon \in (0,1)$. Then, there exists $c(d,p) > 0$ such that for any large enough n, and every x satisfying $\|x\|_\infty \leq \exp\left(n^{(1-\epsilon)\frac{d}{d-1}}\right)$, we have

$$\mathrm{sep}^x_{\mathcal{C}_\infty}(n) \geq cn^{\frac{d-1}{d}}.$$

Thank you !